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# Binding through coupling to a radiation field (Spectral and Scattering Theory and Related Topics)

AUTHOR(S):

Hiroshima, Fumio

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# Binding through coupling to a radiation field

Fumio Hiroshima (廣島 文生)

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## 1 Introduction

### 1.1 Definition

This is a joint work with H. Spohn<sup>1</sup>. We consider a system of one electron interacting with a quantized radiation field. In particular we investigate the so called *Pauli-Fierz* [13] model<sup>2</sup>. Although the Pauli-Fierz model is a nonrelativistic model, it correctly describes the interaction between low energy electrons and photons in a sense. Actually the Lamb shift and gyromagnetic ratio shift were described by using the Pauli-Fierz model. See [2, 14, 12].

In this paper we take the dipole approximation for simplicity. Moreover we suppose that the electron is spinless, moves in the  $d$ -dimensional space, and has the  $d - 1$  transverse degrees of freedom. Throughout this paper we assume

$$d \geq 3.$$

The Hamiltonian of the system is of the form

$$H(\alpha) = \frac{1}{2m} (p \otimes I - \alpha I \otimes A)^2 + V \otimes I + I \otimes H_f \quad (1.1)$$

acting on the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\text{EM}}.$$

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<sup>1</sup> Zentrum Mathematik, Technische Universität München, Archis straße 21, D-80290, München, Germany. e-mail: spohn@mathematik.tu-muenchen.de

<sup>2</sup> See [10] for recent advances of the Pauli-Fierz model.

Here  $\mathcal{F}_{\text{EM}}$  denotes the Boson Fock space over  $W := \oplus^{d-1} L^2(\mathbb{R}^d)$

$$\mathcal{F}_{\text{EM}} := \oplus_{n=0}^{\infty} [\otimes_s^n W],$$

where  $\otimes_s^n W$  denotes the  $n$ -fold symmetric tensor product of  $W$  with  $\otimes_s^0 W := \mathbb{C}$ .  $m$  is the bare mass of the electron and  $\alpha$  a coupling constant. We adopt the unit  $\hbar = 1 = c$ . Then  $\alpha \approx \sqrt{137}$ .  $p = -i\vec{\nabla}$  is the momentum operator canonically conjugate to the position operator  $x$  in  $L^2(\mathbb{R}^d)$ , and  $V = V(x)$  an external potential for which precise conditions will be specified below. The smeared radiation field is defined by

$$A_\mu := \sum_{r=1}^{d-1} \frac{1}{\sqrt{2}} \int e_\mu^r(k) \left\{ \frac{\hat{\varphi}(-k)}{\sqrt{(2\pi)^d \omega(k)}} a^{\dagger r}(k) + \frac{\hat{\varphi}(k)}{\sqrt{(2\pi)^d \omega(k)}} a^r(k) \right\} d^d k,$$

and the free Hamiltonian by

$$H_f := \sum_{r=1}^{d-1} \int \omega(k) a^{\dagger r}(k) a^r(k) dk,$$

where the dispersion relation is given by

$$\omega(k) := |k|.$$

$a^{\dagger r}(k)$  and  $a^r(k)$  denote the annihilation and creation operators, respectively. They satisfy the canonical commutation relations,

$$[a^r(k), a^{\dagger s}(k')] = \delta_{rs} \delta(k - k'), \quad [a^r(k), a^s(k')] = [a^{\dagger r}(k), a^{\dagger s}(k')] = 0.$$

The vectors,  $e^r(k) = (e_1^r(k), \dots, e_d^r(k))$ ,  $r = 1, \dots, d-1$ , denote polarization vectors satisfying

$$e^r(k) \cdot e^s(k) = \delta_{rs}, \quad k \cdot e^r(k) = 0.$$

Finally  $\hat{\varphi}$  denotes a form factor serving as an ultraviolet cutoff. We assume that

$$\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d), \tag{1.2}$$

and

$$\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k). \tag{1.3}$$

(1.2) and (1.3) ensure that  $H(\alpha)$  is a well defined symmetric operator in  $\mathcal{H}$ . It is known that

$$\text{Spec}(H_f) = [0, \infty)$$

and

$$\text{Spec}_p(H_f) = \{0\}.$$

The multiplicity of  $\{0\}$  is one, and

$$H_f \Omega = 0,$$

where  $\Omega := 1 \oplus 0 \oplus 0 \oplus \dots$  is the Fock vacuum in  $\mathcal{F}_{\text{EM}}$ .

## 1.2 Problems

Suppose that  $V$  is relatively bounded with respect to  $-\Delta$  with a sufficiently small relative bound. Then it is proven [8] that  $H(\alpha)$  is self-adjoint on  $D(\Delta \otimes I) \cap D(I \otimes H_f)$  and bounded from below for arbitrary couplings. Moreover by investigating the integral kernel of  $e^{-tH(\alpha)}$ ,  $t \geq 0$ , the uniqueness of the ground state, if it exists, is established in [6]<sup>3</sup>.

In the case when  $-\frac{1}{2m}\Delta + V$  has the positive spectral gap,

$$\inf \text{Spec}_{\text{ess}}(-\frac{1}{2m}\Delta + V) - \inf \text{Spec}(-\frac{1}{2m}\Delta + V) > 0,$$

the existence of the ground state of the full Pauli-Fierz Hamiltonian is established in [3, 5, 9, 4]. In particular, Bach, Fröhlich and Sigal [3] proved it under *no* assumption of infrared cutoff condition<sup>4</sup> but sufficiently weak couplings. For arbitrary couplings, it is established in [4] due to Griesemer, Lieb and Loss.

The main purpose of this paper is to prove the existence of the ground state of  $H(\alpha)$  under *no* assumption of the positive spectral gap. In the

<sup>3</sup> For the *full* Pauli-Fierz Hamiltonian, self-adjointness and the uniqueness of the ground state are established in [8] and [6], respectively.

<sup>4</sup> The condition  $\int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 / \omega(k)^3 dk < \infty$  is called the *infrared cutoff* condition. In the case of  $d = 3$  this condition implies  $0 = \hat{\varphi}(0) = (2\pi)^{-3/2} \int \varphi(x) dx$ , i.e., physically the electron charge turns out to be neutral.

zero spectral gap case,  $-\frac{1}{2m}\Delta + V$  may have no ground state. That is, we show that strong couplings produce the ground state. The physical reasoning behind such a result is as follows. As the electron binds photons it acquires the effective mass

$$m \rightarrow m + \delta m(\alpha^2)$$

which is increasing in  $|\alpha|$ . Roughly speaking  $H(\alpha)$  may be replaced by

$$H(\alpha) \sim -\frac{1}{2(m + \delta m(\alpha^2))}\Delta + V, \quad (1.4)$$

and, for the sufficiently large  $|\alpha|$ , the right hand side of (1.4) may have ground states. Needless to say (1.4) has no sharp mathematical meaning, we show, however, the associated phenomena in this paper.

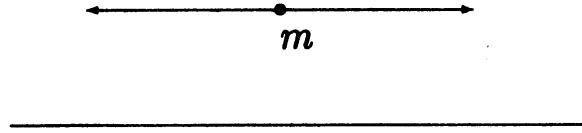


Figure 1:  $H(0)$

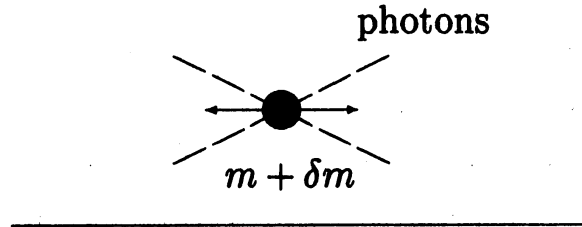


Figure 2:  $H(\alpha)$

This paper is organized as follows. In Section 2 we prove the binding. In Section 3 we give some examples of the external potentials. Finally in Section 4 we give some remarks.

## 2 Binding

We suppose the following assumptions on  $V$ .

- (1)  $\|Vf\| \leq a\|\Delta f\| + b\|f\|$  for  $f \in D(\Delta)$  with sufficiently small  $a \geq 0$ , and positive  $b \geq 0$ .
- (2)  $V \in C^1(\mathbb{R}^d)$  and  $\partial_\mu V \in L^\infty(\mathbb{R}^d)$ ,  $\mu = 1, \dots, d$ .
- (3) There exist  $\mu_0 \geq 1$  and  $r_0 > 0$  such that for all  $\eta > \mu_0$

$$\inf \text{Spec}\left(-\frac{1}{2m}\Delta + \eta V\right) \leq -r_0,$$

and

$$\text{Spec}_{\text{ess}}\left(-\frac{1}{2m}\Delta + \eta V\right) = [0, \infty).$$

It is of interest to investigate sufficiently shallow external potentials. Since  $d \geq 3$ , for such a shallow  $V$ ,  $-\frac{1}{2m}\Delta + V$  may have no ground state. If  $-\frac{1}{2m}\Delta + V$  has no ground state, then the decoupled Hamiltonian

$$H(\alpha = 0) = \left(-\frac{1}{2m}\Delta + V\right) \otimes I + I \otimes H_{\text{f}}$$

also has no ground state.

For later use we define the dilatation unitary of  $L^2(\mathbb{R}^d)$  by

$$D(\kappa)f(k) := \kappa^{d/2}f(k/\kappa),$$

where  $\kappa > 0$  denotes the scaling parameter. The scaled Hamiltonian is defined by

$$\begin{aligned} & H(\alpha, \kappa) \\ & := \kappa^2 D(\kappa)^{-1} \left\{ \frac{1}{2m}(p \otimes I - \alpha I \otimes A)^2 + I \otimes H_{\text{f}} + \frac{1}{\kappa^2} V(x/\kappa) \otimes I \right\} D(\kappa) \end{aligned}$$

$$= \frac{1}{2m}(p \otimes I - \kappa \alpha I \otimes A)^2 + V \otimes I + \kappa^2 I \otimes H_f.$$

We suppose the following technical assumptions on  $\hat{\varphi}$ .

- (1)  $\hat{\varphi}(k) = \hat{\varphi}(|k|)$ .
- (2)  $\omega^{n/2} \hat{\varphi} \in L^2(\mathbb{R}^d)$  for  $n = -5, -4, -3, -2, -1, 0, 1, 2$ .
- (3)  $|\hat{\varphi}(\sqrt{s})|s^{(d-1)/2} \in L^\epsilon([0, \infty), ds)$ ,  $0 < \epsilon < 1$ , and is Lipschitz continuous of order strictly less than one.
- (4)  $\|\hat{\varphi}\omega^{(d-2)/2}\|_\infty < \infty$  and  $\|\hat{\varphi}\omega^{(d-1)/2}\|_\infty < \infty$ .
- (5)  $\hat{\varphi}(k) \neq 0$  for all  $k \neq 0$ .

Thus (1)–(5) ensure the following lemmas<sup>5</sup>.

**Lemma 2.1** *There exist the unitary operator  $U(\kappa)$  such that*

$$U(\kappa)^{-1}H(\alpha, \kappa)U(\kappa) = H_{\text{eff}} + \kappa^2 H_f + \kappa^2 \alpha^2 g + \delta V,$$

where

$$H_{\text{eff}} := -\frac{1}{2m_{\text{eff}}}\Delta + V,$$

$$m_{\text{eff}} = m_{\text{eff}}(\alpha^2) := m + \alpha^2 \left( \frac{d-1}{d} \right) \|\hat{\varphi}/\omega\|^2,$$

and

$$g := \frac{d-1}{2\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\hat{\varphi}/(t^2 + \omega^2)\|^2}{m + \alpha^2 \left( \frac{d-1}{d} \right) \|\hat{\varphi}/\sqrt{t^2 + \omega^2}\|^2} dt.$$

Moreover

$$\delta V = \delta V(\alpha, \kappa) := U(\kappa)^{-1}(V \otimes I)U(\kappa) - V \otimes I.$$

**Lemma 2.2** *We have*

$$-\frac{D(\alpha)}{\kappa}(H_f + I) \leq \delta V \leq \frac{D(\alpha)}{\kappa}(H_f + I)$$

in the sense of form, where  $D(\alpha)$  is a real number satisfying

$$\lim_{|\alpha| \rightarrow \infty} D(\alpha) = 0.$$

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<sup>5</sup> See [1, 11] for details.

Let

$$\alpha_{\text{critical}} := \sqrt{m(\mu_0 - 1)} \sqrt{\frac{d}{d-1}} \|\hat{\varphi}/\omega\|^{-1}.$$

We see

$$H_{\text{eff}} = \frac{m}{m_{\text{eff}}} \left( -\frac{1}{2m} \Delta + \frac{m_{\text{eff}}}{m} V \right).$$

Then, in the case of  $|\alpha| > \alpha_{\text{critical}}$ , it follows that

$$\inf_x V(x) \leq \inf \text{Spec}(H_{\text{eff}}) < -r_0 \frac{m}{m_{\text{eff}}}.$$

In particular the ground states of  $H_{\text{eff}}$  exist.

**Theorem 2.3** *Let  $\kappa = 1$ . There exists  $\alpha_* > \alpha_{\text{critical}}$  such that for all  $|\alpha| \geq \alpha_*$  the ground state of  $H(\alpha)$  exists and it is unique.*

*Proof:* Let  $N$  be the number operator in  $\mathcal{F}_{\text{EM}}$  and  $0 < \nu$ . By a momentum lattice approximation we see that  $H(\alpha) + \nu N$  has the normalized ground state  $\Phi_\nu$ . Let  $E_I$  denote the spectral projection of  $H_{\text{eff}}$  to  $I \subset \mathbb{R}$  and  $P_\Omega$  the projection to  $\Omega$ . Let  $P = E_{(-\infty, -r_0 m/m_{\text{eff}})} \otimes P_\Omega$  and  $\Sigma := \inf \text{Spec}(H_{\text{eff}})$ . Then we can see that

$$(\Phi_\nu, P\Phi_\nu) \geq 1 - \left( \frac{|\alpha|\epsilon}{m_{\text{eff}}} \right)^2 - \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2} \quad (2.1)$$

with some constant  $\epsilon$ . Note that

$$\lim_{|\alpha| \rightarrow 0} \frac{|\alpha|}{m_{\text{eff}}(\alpha^2)} = 0$$

and

$$\lim_{|\alpha| \rightarrow 0} \Sigma = \inf_x V(x).$$

Thus for sufficiently large  $|\alpha|$  the right hand side of (2.1) is strictly positive. Take a subsequence  $\nu'$  such that  $\Phi_{\nu'} \rightarrow \Phi$  as  $\nu \rightarrow 0$  weakly. Since  $P$  is a finite rank operator,  $P\Phi_\nu$  strongly converges to  $P\Phi$  and

$$(\Phi, P\Phi) \geq 1 - \left( \frac{|\alpha|\epsilon}{m_{\text{eff}}} \right)^2 - \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2}$$



holds. In particular  $\Phi \neq 0$ . Hence  $\Phi$  is the ground state.  $\square$

By the assumptions  $H_{\text{eff}}$  has ground states for  $|\alpha| > \alpha_{\text{critical}}$ . We have to make sure that  $H(\alpha)$  has the same properties.

**Theorem 2.4** *We suppose that  $\kappa$  is sufficiently large. Set  $V_\kappa(x) := \kappa^{-2}V(x/\kappa)$ . Then the ground state of*

$$H_\kappa(\alpha) = \frac{1}{2m}(p \otimes I - \alpha I \otimes A)^2 + V_\kappa \otimes I + I \otimes H_f$$

*exists for all  $|\alpha| > \alpha_{\text{critical}}$  and it is unique.*

*Proof:* We have

$$H_\kappa(\alpha) = \frac{1}{\kappa^2} D(\kappa) H(\alpha, \kappa) D(\kappa)^{-1}.$$

Thus it is enough to prove the existence of the ground states of  $H(\alpha, \kappa)$ . From the momentum lattice approximation we see that  $H(\alpha, \kappa) + \nu N$  has the ground state  $\Phi_\nu$ . Moreover we have the inequality

$$(\Phi_\nu, P\Phi_\nu) \geq 1 - \frac{1}{\kappa^6} \left( \frac{|\alpha|\epsilon}{m_{\text{eff}}} \right)^2 - \frac{1}{\kappa^2} \left( \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2} \right).$$

Then the theorem follows in the same way as in Theorem 2.3.  $\square$

### 3 Example

Suppose that

$$V(x) \leq 0.$$

Let

$$N(V) := a_d \int_{\mathbb{R}^d} |mV(x)|^{d/2} dx,$$

where  $a_d$  is a universal constant. The following is known as the Lieb-Thirring equality

$$N(V) = \#\{\text{the nonnegative eigenvalues of } -\frac{1}{2m}\Delta + V\}.$$

Suppose that

$$N(V) < 1.$$

Then  $H(0)$  has no ground state,  $H(\alpha)$  for sufficiently large  $|\alpha|$ , however, has the ground state and it is unique by Theorem 2.3.

**Remark 3.1** *If  $-\frac{1}{2m}\Delta + V$  has the ground state with a positive spectral gap, then  $H(\alpha)$  has the ground state for arbitrary  $\alpha \in \mathbb{R}$ .*

## 4 Concluding remarks

- (1) The full Pauli-Fierz Hamiltonian is defined by

$$H(\alpha) = \frac{1}{2m}(p \otimes I - \alpha A)^2 + V \otimes I + I \otimes H_f.$$

Here under the identification  $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F}_{\text{EM}} dx$

$$A_\mu := \int_{\mathbb{R}^d}^{\oplus} A_\mu(x) dx,$$

and

$$A_\mu(x) := \sum_{r=1}^{d-1} \frac{1}{\sqrt{2}} \times \\ \times \int e_\mu^r(k) \left\{ \frac{\hat{\varphi}(-k)}{\sqrt{(2\pi)^d \omega(k)}} e^{-ikx} a^{\dagger r}(k) + \frac{\hat{\varphi}(k)}{\sqrt{(2\pi)^d \omega(k)}} e^{ikx} a^r(k) \right\} d^d k.$$

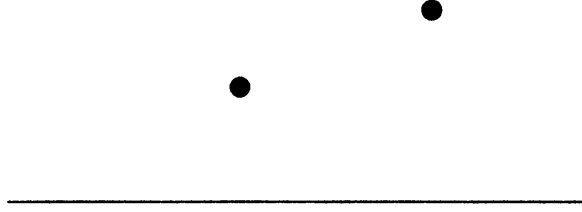
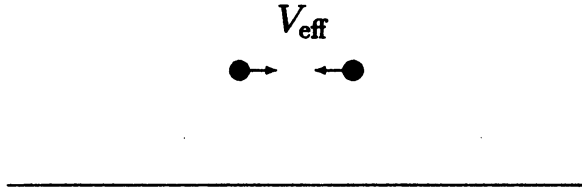
For the full Pauli-Fierz Hamiltonian, it seems to be unknown the binding.

- (2) For  $\alpha$  such that  $0 < |\alpha| < \alpha_{\text{critical}}$ , no existence of the ground state is not known.
- (3) The Nelson Hamiltonian with two charged particles is defined by

$$H_{\text{Nelson}}(\alpha) := \left( -\frac{1}{2m}\Delta + V \right) \otimes I + I \otimes H_N + \alpha \phi$$

acting on

$$\mathcal{H} := L^2(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathcal{F},$$

Figure 3:  $H_{\text{Nelson}}(0)$ Figure 4:  $H_{\text{Nelson}}(\alpha)$ 

where  $\mathcal{F}$  denotes the Boson Fock space over  $L^2(\mathbb{R}^d)$ . The free Hamiltonian is defined by

$$H_N := \int \omega(k) a^\dagger(k) a(k) dk$$

and the scalar field by

$$\phi := \int_{\mathbb{R}^d \times \mathbb{R}^d}^\oplus \phi(x) dx,$$

$$\phi(x) := \sum_{j=1}^2 \frac{1}{\sqrt{2}} \int \hat{\lambda}(-k) e^{-ikx^j} a^\dagger(k) + \hat{\lambda}(k) e^{ikx^j} a(k) dk.$$

Roughly speaking  $H_{\text{Nelson}}(\alpha)$  may be replaced by

$$H_{\text{Nelson}}(\alpha) \sim -\frac{1}{2m} \Delta + V + V_{\text{eff}},$$

$$V_{\text{eff}}(x^1, x^2) = -\frac{\alpha^2}{2} \int_{\mathbb{R}^d} \frac{\hat{\lambda}(k)^2}{\omega(k)} e^{-ik(x^1 - x^2)} dk.$$

Then we can also prove the binding of the Nelson Hamiltonian under certain conditions. We omit details.

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